Robust Distributed Estimation: Extending Gossip Algorithms to Ranking and Trimmed Means

Can we design more robust gossip algorithms?

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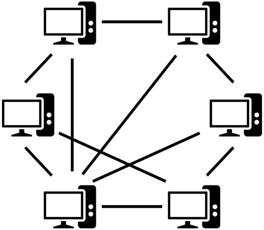
Gossip Algorithms: Where and Why?

- When data is distributed over a network.
- If we have resource and communication constraints.
- If we don't have access to a central server.

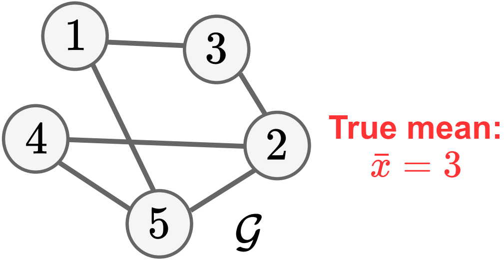


Peer-to-Peer (P2P) Networks

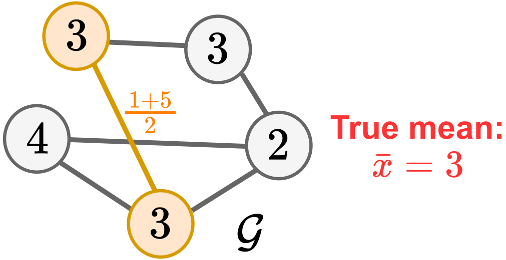
Wireless Sensor Networks

...and many more!

Simple Example: Gossip for Averaging



Select random $(i, j) \in E$, average X_i and X_j



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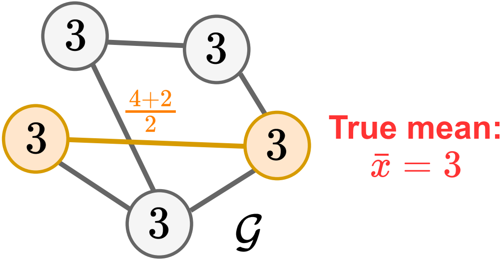

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What Makes the Theoretical Analysis Work?

- Averaging matrix: $t > 0$, for a random edge $(i, j) \in E$,

$$\mathbf{W}_2(t) = \mathbf{I}_n - \frac{1}{2}\left(\mathbf{e}_i - \mathbf{e}_j\right)\left(\mathbf{e}_i - \mathbf{e}_j\right)^\top .$$

- Its expectation with respect to the edge sampling process:

$$\mathbf{W}_2 = \mathbb{E}\left[\mathbf{W}_2(t)\right] = \mathbf{I}_n - \underbrace{\mathbf{L}}_{\text{Graph Laplacian}}/(2|E|).$$

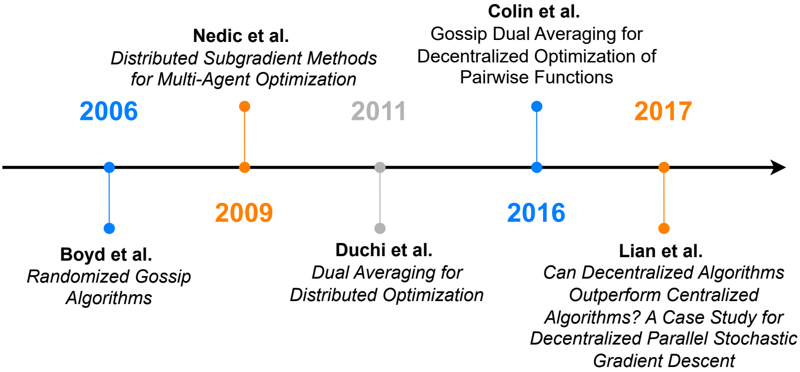
- Properties: symmetric, doubly stochastic and

$$\left\|\mathbf{W}_2 - \mathbf{1}_n\mathbf{1}_n^\top/n\right\|_{\mathrm{op}} \leq 1 - \underbrace{c}_{\text{Graph's connectivity}}/2 < 1.$$

- Convergence in expectation:

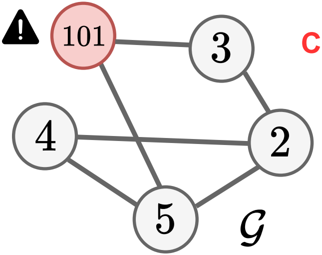
$$\|\mathbb{E}[\mathbf{Y}(t)] - \bar{x}_n\mathbf{1}_n\| \leq \left(1 - \frac{c}{2}\right)^t \|\mathbf{X} - \bar{x}_n\mathbf{1}_n\| .$$

Applications of Gossip for Decentralized Learning



When Gossip Goes Wrong...

- What if a node's value is corrupted? All estimates will be corrupted!
- The average is not robust to outliers. We need more robust statistics, e.g., trimmed mean (see Huber's robustness).



Corrupted mean:
$\bar{x} = 23$

True mean:
$\bar{x} = 3$

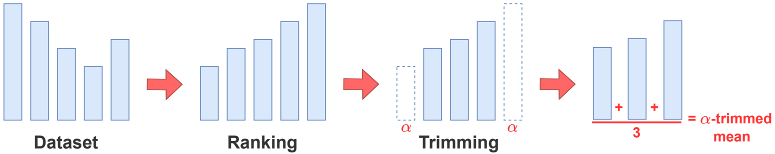
Robust Statistics: Trimmed Means

- Robust average: denoting $m = \lfloor n\alpha \rfloor$,

Order statistic

$$\underbrace{\bar{x}_\alpha}_{\alpha\text{-trimmed mean}} = \frac{1}{n-2m} \sum_{k=m+1}^{n-m} X_{n(k)}$$

- Breakdown point is equal to α.



Problem Formulation and Framework

- Communication network: connected undirected graph $G = (V, E)$ with n nodes. Each node k holds X_k.
- Huber's contamination model: fraction ε of corrupted nodes.
- Synchronous/asynchronous setting: global/local iteration counter.

Using ranks, we can compute a robust average:

$$
\begin{aligned}
\underbrace{\bar{x}_\alpha}_{\alpha\text{-trimmed mean}} &= \frac{1}{n-2m} \sum_{k=m+1}^{n-m} X_{n(k)} \\
&= \frac{1}{n} \sum_k f\left(\underbrace{r_k}_{\text{Rank}}\right) \cdot \underbrace{X_k}_{\text{Observation}}
\end{aligned}
$$

GoRank – A Gossip Algorithm for Ranking (Asynchronous)

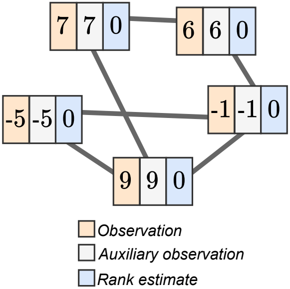
$$r_k = 1 + n\left(\frac{1}{n}\sum_{l=1}^{n}\mathbb{I}_{\{X_k > X_l\}}\right) = 1 + nr'_k$$

Rank of node k

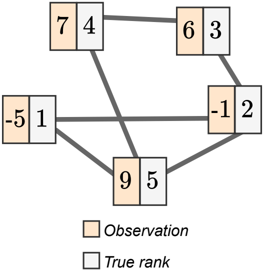
# of nodes

Pairwise comparison

Estimates:

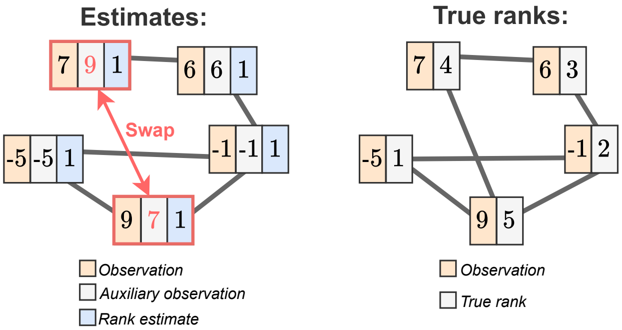


True ranks:



GoRank – A Gossip Algorithm for Ranking (Asynchronous)

$$\underbrace{R_k(t)}_{\text{Rank estimate}} = 1 + n\left(\frac{1}{\underbrace{C_k(t)}_{\text{Counter}}}\sum_{s=1}^{C_k(t)} \mathbb{I}_{\{X_k > \underbrace{Y_k(s)}_{\text{Auxiliary observation}}\}}\right) = 1 + nR'_k(t)$$



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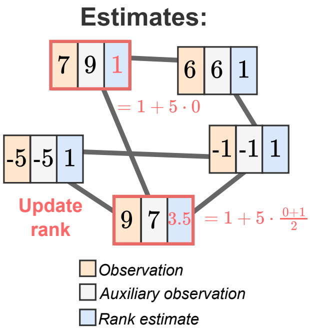
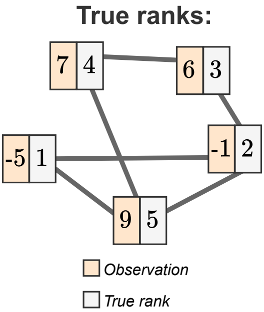




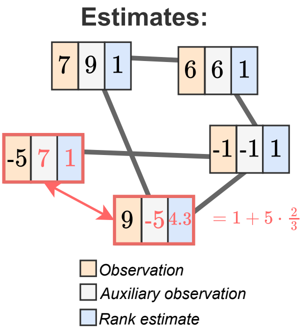
GoRank – A Gossip Algorithm for Ranking (Asynchronous)

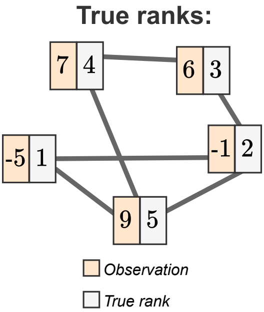
$$\boxed{R_k(t)} = 1 + n\left(\frac{1}{\boxed{C_k(t)}}\sum_{s=1}^{C_k(t)} \mathbb{I}_{\{X_k > \boxed{Y_k(s)}\}}\right) = 1 + nR'_k(t)$$

Rank estimate

Counter

Auxiliary observation





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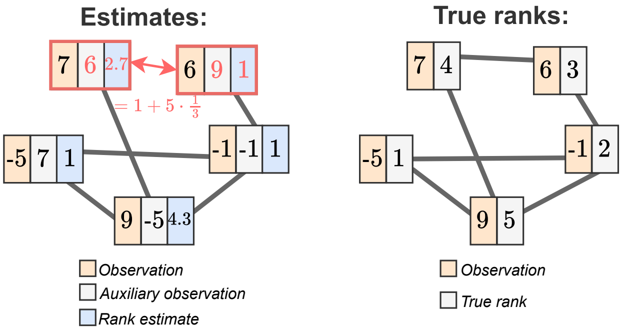

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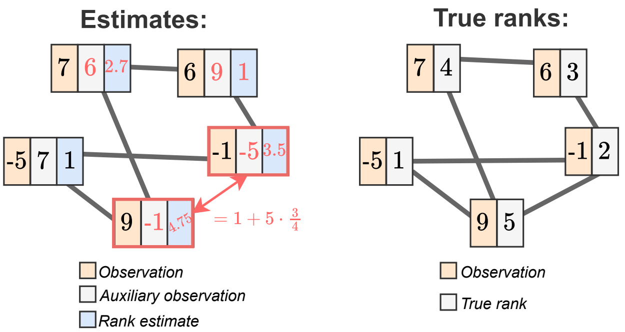

GoRank Shows Great Empirical Performance

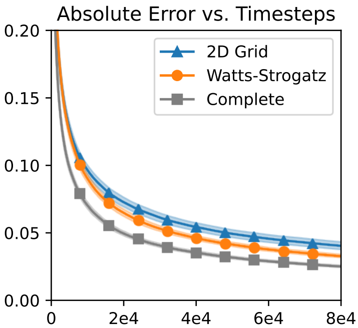

Figure 1: Graph topology has an impact on convergence.

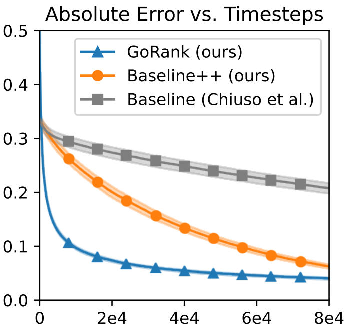

Figure 2: GoRank outperforms previous work in large graphs.

GoRank: Theoretical Analysis

	GoRank (synchronous)
Rank Estimator of Node k	$1 + n \cdot \frac{1}{t} \sum_{s=1}^{t} \mathbb{I}_{\{X_k > Y_k(s)\}}$
Convergence in Expectation (Theorem 1)	$\mathcal{O}\left(\frac{\sigma_k}{ct}\right)$
Expected Abs. Error (Theorem 2)	$\mathcal{O}\left(\frac{\sigma_k}{\sqrt{ct}}\right)$

Table 1: Convergence Rate Bounds.

σ_k = standard deviation

c = spectral gap of the Laplacian

"The absolute error converges in $\mathcal{O}\left(\frac{1}{\sqrt{t}}\right)$."

GoRank: What Makes the Theoretical Analysis Work?

- Swapping matrix: $t > 0$, for a random edge $(i, j) \in E$,

$$\boxed{\mathbf{W}_1(t) = \mathbf{I}_n - (\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^\top .}$$

- Its expectation with respect to the edge sampling process:

$$\mathbf{W}_1 = \mathbb{E}[\mathbf{W}_1(t)] = \mathbf{I}_n - \boxed{\mathbf{L}}/|E|.$$

Graph Laplacian

- Properties: symmetric, doubly stochastic and

$$\left\|\mathbf{W}_1 - \mathbf{1}_n\mathbf{1}_n^\top/n\right\|_{\mathrm{op}} \leq 1 - \boxed{c} < 1.$$

Graph's connectivity

GoTrim – A Gossip Algorithm for Trimmed Means

- Computes a weighted average using current ranks:

$$\frac{1}{n}\sum_{k} f(r_k) \cdot X_k.$$

- **Idea 1**: Perform standard (weighted) gossip-based averaging using current ranks:

$$\text{weight} = f(r_k).$$

- **Idea 2**: Since ranks vary, correct past errors by injecting:

$$(\text{weight}^{\text{new}} - \text{weight}^{\text{prev}}) \cdot X_k\,.$$

GoTrim Shows Great Empirical Performance

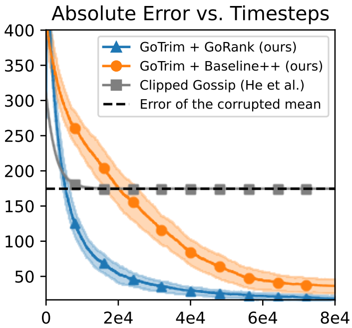

Figure 3: Simulated Dataset

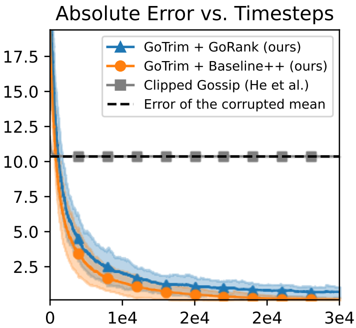

Figure 4: Basel Luftklima Dataset

"GoTrim quickly improves over the corrupted mean."

GoTrim: Theoretical Analysis

	GoTrim (synchronous)
Trimmed Mean Estimator of Node k	$\frac{1}{n}\sum_{k=1}^{n} f(R_k) \cdot X_k$
Convergence in Expectation (Theorem 3)	$\mathcal{O}\left(\frac{1}{c^2 t}\right)$
τ-Breakdown Point (Theorem 4)	$\approx \alpha - \frac{1}{\sqrt{t-T}}$

Table 2: Convergence Rate Bounds.

c = spectral gap of the Laplacian

"The bias converges in $\mathcal{O}\left(\frac{1}{t}\right)$."

Conclusion and Future Work

- We design novel gossip algorithms for ranking (GoRank) and trimmed means estimation (GoTrim).
- We show strong empirical performance and have solid theoretical guarantees.
- **Future work**:
  - Extension to multivariate data
  - Decentralized robust optimization
  - Byzantine Robustness

Our paper:

References

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